On spanning tree congestion of Hamming graphs

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Abstract

We present a tight lower bound for the spanning tree congestion of Hamming graphs.

1 Preliminaries

The spanning tree congestion of graphs has been studied intensively [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. In this note, we study the spanning tree congestion of Hamming graphs. We present a lower bound for the spanning tree congestion of Hamming graphs. That is, in our terminology, we show that $\mathrm{stc}(K_n^d) \geq \frac{1}{d}(n^d - 1)\log_n d$. It is known that $\mathrm{stc}(K_n^d) = O\left(\frac{1}{d} n^d \log_n d\right)$ [10]. Thus our lower bound is asymptotically tight.

For a graph G, we denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph induced by S. For an edge $e \in E(G)$, we denote by $G - e$ the graph obtained from G by deleting e. Let $N_G(v)$ denote the neighborhood of $v \in V(G)$ in G; that is, $N_G(v) = \{u \mid \{u, v\} \in E(G)\}$. We denote the degree of a vertex $v \in V(G)$ by $\deg_G(v)$, and the maximum degree of G by $\Delta(G)$; that is, $\deg_G(v) = |N_G(v)|$ and $\Delta(G) = \max_{v \in V(G)} \deg_G(v)$. A graph G is *r-regular* if $\deg_G(v) = r$ for every $v \in V(G)$.

For $S \subseteq V(G)$, we denote the edge set of $G[S]$ by $\iota_G(S)$, and the *boundary edge set* by $\theta_G(S)$; that is, $\iota_G(S) = \{\{u, v\} \in E(G) \mid u, v \in S\}$ and $\theta_G(S) = \{\{u, v\} \in E(G) \mid \text{exactly one of } u, v \text{ is in } S\}$. We define the function ι and θ also for a positive integer $s \leq |V(G)|$ as $\iota_G(s) = \max_{S \subseteq V(G),\ |S|=s} |\iota_G(S)|$ and $\theta_G(s) = \min_{S \subseteq V(G),\ |S|=s} |\theta_G(S)|$. Let T be a spanning tree of a connected graph G. The *congestion* of $e \in E(T)$ as $\mathrm{cng}_G(e) = |\theta_G(L_e)|$, where L_e is the vertex set of one of the two components of $T - e$. The *congestion of T in G*, denoted by $\mathrm{cng}_G(T)$, is the maximum congestion over all edges in T. We define the *spanning tree congestion* of G, denoted by $\mathrm{stc}(G)$, as the minimum congestion over all spanning trees of G.

The *d-dimensional Hamming graph* K_n^d is the graph with vertex set $\{0, \ldots, n-1\}^d$ in which two vertices are adjacent if and only if their corresponding d-dimensional vectors differ in exactly one place. It is evident that K_n^d is $d(n-1)$-regular. The exact value of $\mathrm{stc}(K_n^2)$ is known [6]. Also, $\mathrm{stc}(K_2^d)$ is determined asymptotically [8].

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2 The lower bound

Here, we present the lower bound. We need the following three lemmas.

Lemma 2.1 ([1])**.** *If G is r-regular and $S \subseteq V(G)$, then* $|\theta_G(S)| = r|S| - 2|\iota_G(S)|$.

Lemma 2.2 ([17])**.** *Let G be a subgraph of K_n^d. If G has s vertices and t edges, then* $2t \leq (n-1)s \log_n s$.

Lemma 2.3 ([4, 7])**.** *For any connected graph G,* $\mathrm{stc}(G) \geq \min_{s=\lceil(|V(G)|-1)/\Delta(G)\rceil}^{\lfloor|V(G)|/2\rfloor} \theta(s)$.

Theorem 2.4. $\mathrm{stc}(K_n^d) \geq (n^d - 1)\log_n d/d$ *for* $n, d \geq 3$.

Proof. Since K_n^d is $d(n-1)$-regular, Lemmas 2.1 and 2.2 imply that $\theta_{K_n^d}(s) \geq (n-1)s(d - \log_n s)$. Let $f(s) = (n-1)s(d - \log_n s)$ and $f'(s)$ be the derived function of $f(s)$. Then $f'(s) = (n-1)(d - 1/\ln n - \log_n s)$, and thus, $f(s)$ is increasing for $(n^d - 1)/(d(n-1)) \leq s \leq n^{d-1/\ln n}$ and decreasing for $n^{d-1/\ln n} \leq s \leq n^d/2$. Therefore,

$$\begin{aligned}
\min_{s=\lceil(n^d-1)/(d(n-1))\rceil}^{\lfloor n^d/2\rfloor} f(s) &= \min\left\{ f\left(\left\lceil \frac{n^d-1}{d(n-1)} \right\rceil\right), f\left(\left\lfloor \frac{n^d}{2} \right\rfloor\right)\right\} \\
&\geq \min\left\{ f\left(\frac{n^d-1}{d(n-1)}\right), f\left(\frac{n^d}{2}\right)\right\} \\
&= \min\left\{ \frac{n^d-1}{d}\left(d - \log_n \frac{n^d-1}{d(n-1)}\right), \frac{(n-1)n^d}{2}\left(d - \log_n \frac{n^d}{2}\right)\right\} \\
&\geq \min\left\{ \frac{n^d-1}{d}\log_n d, \frac{(n-1)n^d}{2}\log_n 2\right\}.
\end{aligned}$$

Thus, by Lemma 2.3, it holds that

$$\mathrm{stc}(K_n^d) \geq \min\left\{ \frac{n^d-1}{d}\log_n d, \frac{(n-1)n^d}{2}\log_n 2\right\}.$$

By a simple calculation, we can see that $\frac{n^d-1}{d}\log_n d \leq \frac{(n-1)n^d}{2}\log_n 2$ for $d = 2, 3$. Since $n^d - 1 \leq (n-1)n^d$ and $(\log_n d)/d \leq (\log_n 2)/2$ for $d \geq 4$, the theorem holds. □

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